Roll No. ☐☐☐☐☐☐☐☐

**Q.P. Code 65/1/1**

# MATHEMATICS

CBSE Class XII Board Examination 2024 — with answers

**Time allowed: 3 hours** **Maximum Marks: 80**

**General Instructions:**

1. The paper has 38 questions in five sections (A to E); all of them must be answered.
2. Section A: Q1–18 are multiple choice and Q19–20 are Assertion-Reason, 1 mark each.
3. Section B: Q21–25, very short answer (VSA), 2 marks each.
4. Section C: Q26–31, short answer (SA), 3 marks each.
5. Section D: Q32–35, long answer (LA), 5 marks each.
6. Section E: Q36–38, case-study based, 4 marks each.
7. There is no overall choice, but internal choices (OR) are given in 2 questions of Section B, 3 of Section C, 2 of Section D and 2 of Section E. Calculators are not allowed.

## SECTION A

*Q1–20 carry 1 mark each (Q1–18 MCQ; Q19–20 Assertion-Reason).*

**1.** The function $f : \mathbb{R}_+ \to \mathbb{R}$ (where $\mathbb{R}_+$ is the set of all non-negative real numbers) is given by $f(x) = 4x + 3$. Then f is: **1**

**(A) one-one but not onto**

**(B)** onto but not one-one

**(C)** both one-one and onto

**(D)** neither one-one nor onto

**ANSWER**

(A) one-one but not onto

One-one: $4x_1 + 3 = 4x_2 + 3 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: for x ≥ 0, f(x) ≥ 3. So values like 0 or −5 in the codomain $\mathbb{R}$ are never reached. Range = $[3, \infty) \neq \mathbb{R}$, so f is not onto.

Common mistake: forgetting that the domain is only x ≥ 0 and the codomain is all of $\mathbb{R}$.

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**2.** A matrix has 36 elements. How many different orders can it have? **1**

**(A)** 13

**(B)** 3

**(C)** 5

**(D) 9**

**ANSWER**

(D) 9

An order m × n needs mn = 36, so each order corresponds to a divisor m of 36.

Divisors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36 → 9 divisors.

Orders: 1×36, 2×18, 3×12, 4×9, 6×6, 9×4, 12×3, 18×2, 36×1 = 9 orders.

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**3.** Consider $f(x) = \begin{cases} x^2 + 3, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Which statement is true? **1**

**(A)** f(x) is continuous and differentiable for all x ∈ $\mathbb{R}$

**(B)** f(x) is continuous for all x ∈ $\mathbb{R}$

**(C) f(x) is continuous and differentiable for all x ∈ $\mathbb{R}$ − {0}**

**(D)** f(x) is discontinuous at infinitely many points

**ANSWER**

(C) continuous and differentiable on $\mathbb{R} - \{0\}$

At x = 0: $\lim_{x \to 0}(x^2 + 3) = 3$ but f(0) = 1, so f is discontinuous (hence not differentiable) at 0.

For x ≠ 0, f(x) = x² + 3 is a polynomial, so it is continuous and differentiable there.

So f is continuous and differentiable on $\mathbb{R} - \{0\}$, and has only one point of discontinuity.

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**4.** f(x) is continuous on [a, b] and differentiable on (a, b). f(x) is strictly increasing on (a, b) if: **1**

**(A)** f′(x) < 0 for all x ∈ (a, b)

**(B) f′(x) > 0 for all x ∈ (a, b)**

**(C)** f′(x) = 0 for all x ∈ (a, b)

**(D)** f(x) > 0 for all x ∈ (a, b)

**ANSWER**

(B) f′(x) > 0 on (a, b)

Standard result: if f′(x) > 0 at every point of (a, b), f is strictly increasing on (a, b).
f′ < 0 gives strictly decreasing, f′ = 0 gives a constant function, and f(x) > 0 says nothing about increasing.

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**5.** Given $\begin{bmatrix} x+y & 2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, find $\frac{24}{x} + \frac{24}{y}$. **1**

**(A)** 7

**(B)** 6

**(C)** 8

**(D) 18**

**ANSWER**

(D) 18

Equal matrices → equal entries: x + y = 6 and xy = 8.

$$\frac{24}{x} + \frac{24}{y} = \frac{24(x+y)}{xy} = \frac{24 \times 6}{8} = 18.$$

No need to find x and y separately.

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**6.** $\int_a^b f(x)\,dx$ equals: **1**

**(A)** $\int_a^b f(a-x)\,dx$

**(B)** $\int_a^b f(a+b-x)\,dx$

**(C)** $\int_a^b f(x-(a+b))\,dx$

**(D)** $\int_a^b f((a-x)+(b-x))\,dx$

**ANSWER**

(B) $\int_a^b f(a+b-x)\,dx$

Property of definite integrals: put t = a + b − x, dt = −dx; x = a → t = b, x = b → t = a.

$$\int_a^b f(a+b-x)\,dx = \int_b^a f(t)(-dt) = \int_a^b f(t)\,dt.$$

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**7.** θ is the angle between two unit vectors $\hat{a}$ and $\hat{b}$, and sin θ = 3/5. Then $\hat{a} \cdot \hat{b}$ equals: **1**

**(A)** ± 3/5

**(B)** ± 3/4

**(C) ± 4/5**

**(D)** ± 4/3

**ANSWER**

(C) ± 4/5

$\hat{a} \cdot \hat{b} = |\hat{a}||\hat{b}| \cos\theta = \cos\theta$.

cos θ = ±√(1 − 9/25) = ±4/5 (θ may be acute or obtuse).

**8.** Find the integrating factor of $(1 - x^2)\frac{dy}{dx} + xy = ax$, where −1 < x < 1. **1**

**(A)** $\frac{1}{x^2 - 1}$ **(B)** $\frac{1}{\sqrt{x^2 - 1}}$

**(C)** $\frac{1}{1 - x^2}$ **(D)** $\frac{1}{\sqrt{1 - x^2}}$

**ANSWER**
(D) $1/\sqrt{1 - x^2}$

Standard form: $\frac{dy}{dx} + \frac{x}{1 - x^2}\, y = \frac{ax}{1 - x^2}$, so P = x/(1 − x²).
$\int P\, dx = -\frac{1}{2}\log(1 - x^2)$.
IF = $e^{-\frac{1}{2}\log(1-x^2)} = \frac{1}{\sqrt{1 - x^2}}$.

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**9.** A line has direction cosines √3 k, √3 k, √3 k. Find k. **1**

**(A)** ± 1 **(B)** ± √3

**(C)** ± 3 **(D) ± 1/3**

**ANSWER**
(D) k = ± 1/3

Sum of squares of direction cosines = 1: 3k² + 3k² + 3k² = 1 ⇒ 9k² = 1 ⇒ k = ± 1/3.

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**10.** A linear programming problem is about optimising (maximising or minimising) a: **1**

**(A)** logarithmic function **(B) linear function**

**(C)** quadratic function **(D)** exponential function

**ANSWER**
(B) linear function

In an LPP the objective function Z = ax + by is linear, and the constraints are linear inequalities.

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**11.** If P(A | B) = P(A′ | B), which statement is true? **1**

**(A)** P(A) = P(A′) **(B)** P(A) = 2 P(B)

**(C) P(A ∩ B) = ½ P(B)** **(D)** P(A ∩ B) = 2 P(B)

**ANSWER**
(C) P(A ∩ B) = ½ P(B)

P(A | B) + P(A′ | B) = 1. If they are equal, each is ½.
So P(A ∩ B)/P(B) = ½ ⇒ P(A ∩ B) = ½ P(B).

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**12.** Evaluate $\begin{vmatrix} x+1 & x-1 \\ x^2+x+1 & x^2-x+1 \end{vmatrix}$. **1**

**(A)** $2x^3$ **(B) 2**

**(C)** 0 **(D)** $2x^3 - 2$

**ANSWER**
(B) 2

Value = $(x + 1)(x^2 - x + 1) - (x - 1)(x^2 + x + 1)$.
Using $a^3 + b^3$ and $a^3 - b^3$: $= (x^3 + 1) - (x^3 - 1) = 2$.

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**13.** Find the derivative of $\sin(x^2)$ with respect to x at $x = \sqrt{\pi}$. **1**

**(A)** 1 **(B)** −1

**(C)** $\mathbf{-2\sqrt{\pi}}$ **(D)** $\mathbf{2\sqrt{\pi}}$

**ANSWER**
(C) $-2\sqrt{\pi}$

Chain rule: $d/dx \sin(x^2) = 2x \cos(x^2)$.
At $x = \sqrt{\pi}$: $2\sqrt{\pi} \cdot \cos \pi = 2\sqrt{\pi} \cdot (-1) = -2\sqrt{\pi}$.

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**14.** Find the order and degree of $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}$. **1**

**(A)** 1, 2 **(B)** 2, 3

**(C) 2, 1** **(D)** 2, 6

**ANSWER**
(C) order 2, degree 1

Highest derivative present is $d^2y/dx^2$ → order 2.
The equation is already a polynomial in derivatives and $d^2y/dx^2$ appears with power 1 → degree 1.
Common mistake: taking the power 3 (or 6) of dy/dx as the degree; degree is the power of the HIGHEST order derivative.

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**15.** Find the vector whose initial point is B(3, −4, 7) and terminal point is A(2, −3, 5). **1**

**(A)** $\mathbf{\hat{i} - \hat{j} + 2\hat{k}}$ **(B)** $\mathbf{\hat{i} + \hat{j} + 2\hat{k}}$

**(C)** $\mathbf{-\hat{i} - \hat{j} - 2\hat{k}}$ **(D)** $\mathbf{-\hat{i} + \hat{j} - 2\hat{k}}$

**ANSWER**
(D) $-\hat{i} + \hat{j} - 2\hat{k}$

Vector from B to A = (position of A) − (position of B) = $(2 - 3)\hat{i} + (-3 + 4)\hat{j} + (5 - 7)\hat{k} = -\hat{i} + \hat{j} - 2\hat{k}$.

**16.** The distance of the point P(a, b, c) from the y-axis is: **1**

**(A)** b

**(B)** $b^2$

**(C)** $\sqrt{a^2 + c^2}$

**(D)** $a^2 + c^2$

**ANSWER**

(C) $\sqrt{a^2 + c^2}$

Foot of perpendicular from P on the y-axis is (0, b, 0).

Distance = $\sqrt{(a - 0)^2 + (b - b)^2 + (c - 0)^2} = \sqrt{a^2 + c^2}$.

**17.** How many corner points does the feasible region of $x \geq 0$, $y \geq 0$, $x + y \geq 4$ have? **1**

**(A)** 0

**(B)** 1

**(C)** 2

**(D)** 3

**ANSWER**

(C) 2

The region lies above the line $x + y = 4$ in the first quadrant and is unbounded.

Its corner points are where $x + y = 4$ meets the axes: (4, 0) and (0, 4). The origin is not in the region.

So 2 corner points.

**18.** A and B are non-zero square matrices of the same order with $(A + B)^2 = A^2 + B^2$. Then: **1**

**(A)** AB = O

**(B)** AB = −BA

**(C)** BA = O

**(D)** AB = BA

**ANSWER**

(B) AB = −BA

$(A + B)^2 = A^2 + AB + BA + B^2$ (matrix multiplication is not commutative, so AB + BA, not 2AB).

Given it equals $A^2 + B^2 \Rightarrow AB + BA = O \Rightarrow AB = -BA$.

**19.** **1**

Assertion (A): For $A = \begin{bmatrix} 1 & \cos\theta & 1 \\ -\cos\theta & 1 & \cos\theta \\ -1 & -\cos\theta & 1 \end{bmatrix}$, $\theta \in [0, 2\pi]$, we have $|A| \in [2, 4]$.

Reason (R): $\cos\theta \in [-1, 1]$ for all $\theta \in [0, 2\pi]$.

**(A) Both A and R are true, and R is the correct explanation of A.**

**(B)** Both A and R are true, but R is not the correct explanation of A.

**(C)** A is true, but R is false.

**(D)** A is false, but R is true.

**ANSWER**

(A) Both true, R explains A

Let $c = \cos\theta$. Expanding along R1: $|A| = 1(1 + c^2) - c(-c + c) + 1(c^2 + 1) = 2 + 2c^2$.

Since $c \in [-1, 1]$, $c^2 \in [0, 1]$, so $|A| \in [2, 4]$. A is true.

R is true and it is exactly the fact used to get the range of |A|, so R explains A.

**20.** Assertion (A): No line in space can be perpendicular to all three of the x, y and z axes at the same time. **1**

Reason (R): If a line makes angles $\alpha$, $\beta$, $\gamma$ with the positive x, y, z axes, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

**(A) Both A and R are true, and R is the correct explanation of A.**

**(B)** Both A and R are true, but R is not the correct explanation of A.

**(C)** A is true, but R is false.

**(D)** A is false, but R is true.

**ANSWER**

(A) Both true, R explains A

If a line were perpendicular to all three axes, $\alpha = \beta = \gamma = 90°$, so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0$, contradicting R (sum must be 1).

So A is true, R is a standard true identity, and R is the reason A holds.

## SECTION B

*Q21–25 are very short answer (VSA) questions, 2 marks each.*

**21.** Is the function f(x) = x²|x| differentiable at x = 0? Justify. **2**

**ANSWER**

Yes, f is differentiable at x = 0 and f′(0) = 0.

f(x) = x³ for x ≥ 0 and f(x) = −x³ for x < 0; f(0) = 0.

RHD = $\lim_{h\to 0^+} \dfrac{h^3 - 0}{h} = \lim h^2 = 0.$

LHD = $\lim_{h\to 0^+} \dfrac{f(-h) - f(0)}{-h} = \lim_{h\to 0^+} \dfrac{h^3}{-h} = \lim(-h^2) = 0.$

RHD = LHD = 0, so f is differentiable at x = 0.

**OR**

Given $y = \sqrt{\tan\sqrt{x}}$, prove that $\sqrt{x}\,\dfrac{dy}{dx} = \dfrac{1+y^4}{4y}$.

**ANSWER**

Proved: $\sqrt{x}\,y' = (1+y^4)/(4y)$

Chain rule: $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \sec^2\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}.$

So $\sqrt{x}\,\dfrac{dy}{dx} = \dfrac{\sec^2\sqrt{x}}{4\sqrt{\tan\sqrt{x}}} = \dfrac{1+\tan^2\sqrt{x}}{4\sqrt{\tan\sqrt{x}}}.$

Since y² = tan√x, tan²√x = y⁴ and √(tan√x) = y, giving $\dfrac{1+y^4}{4y}$.

**22.** Show that f(x) = 4x³ − 18x² + 27x − 7 has no maximum or minimum value. **2**

**ANSWER**

f′(x) = 3(2x − 3)² ≥ 0, so f is increasing on ℝ: no maxima/minima.

f′(x) = 12x² − 36x + 27 = 3(4x² − 12x + 9) = 3(2x − 3)².

f′(x) ≥ 0 for all x, and f′ = 0 only at x = 3/2, where f′ does not change sign.

So f is increasing on ℝ and has no local maximum or minimum.

Common mistake: calling x = 3/2 an extreme point just because f′(3/2) = 0; check the sign change.

**23.** Find $\int x\sqrt{1+2x}\,dx$. **2**

**ANSWER**

$\dfrac{(1+2x)^{5/2}}{10} - \dfrac{(1+2x)^{3/2}}{6} + C$

Put $1 + 2x = t^2$, so $x = (t^2 - 1)/2$ and $dx = t\,dt$.

Integral $= \int \dfrac{t^2-1}{2} \cdot t \cdot t\,dt = \dfrac{1}{2}\int (t^4 - t^2)\,dt = \dfrac{1}{2}\left(\dfrac{t^5}{5} - \dfrac{t^3}{3}\right) + C.$

Back-substitute $t = \sqrt{(1+2x)}$: $\dfrac{(1+2x)^{5/2}}{10} - \dfrac{(1+2x)^{3/2}}{6} + C.$

**OR**

Evaluate $\displaystyle\int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx.$

**ANSWER**

2

Put $\sqrt{x} = t \Rightarrow dx = 2t\,dt$; limits $x = 0 \to t = 0$, $x = \pi^2/4 \to t = \pi/2$.

Integral $= \int_0^{\pi/2} \dfrac{\sin t}{t} \cdot 2t\,dt = 2\int_0^{\pi/2} \sin t\,dt = 2[-\cos t]_0^{\pi/2} = 2(0+1) = 2.$

**24.** $\vec{a}$ and $\vec{b}$ are non-zero vectors with $(\vec{a} + \vec{b}) \perp \vec{a}$ and $(2\vec{a} + \vec{b}) \perp \vec{b}$. Prove that $|\vec{b}| = \sqrt{2}\,|\vec{a}|$. **2**

**ANSWER**

Proved: $|b|^2 = 2|a|^2 \Rightarrow |b| = \sqrt{2}\,|a|$

Perpendicular ⇒ dot product zero.

(1) $(\vec{a} + \vec{b}) \cdot \vec{a} = 0 \Rightarrow |\vec{a}|^2 + \vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \cdot \vec{b} = -|\vec{a}|^2$.

(2) $(2\vec{a} + \vec{b}) \cdot \vec{b} = 0 \Rightarrow 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 0$.

Substitute (1) in (2): $-2|\vec{a}|^2 + |\vec{b}|^2 = 0 \Rightarrow |\vec{b}|^2 = 2|\vec{a}|^2 \Rightarrow |\vec{b}| = \sqrt{2}|\vec{a}|$.

**25.** ABCD is a parallelogram (figure: AB is the top side, DC the bottom side, and the diagonal DB is drawn from D to B). Given $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, find $\vec{AD}$ and then the area of the parallelogram. **2**

**ANSWER**

$\vec{AD} = -\hat{i} + 2\hat{j} + 3\hat{k}$; area = $11\sqrt{5}$ sq units

Triangle law: $\vec{AD} + \vec{DB} = \vec{AB} \Rightarrow \vec{AD} = (2-3)\hat{i} + (-4+6)\hat{j} + (5-2)\hat{k} = -\hat{i} + 2\hat{j} + 3\hat{k}$.

Area $= |\vec{AD} \times \vec{AB}|$.

$\vec{AD} \times \vec{AB} = \hat{i}(2 \cdot 5 - 3 \cdot (-4)) - \hat{j}((-1) \cdot 5 - 3 \cdot 2) + \hat{k}((-1)(-4) - 2 \cdot 2) = 22\hat{i} + 11\hat{j} + 0\hat{k}$

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Area = $\sqrt{(484 + 121)} = \sqrt{605} = 11\sqrt{5}$ square units.

## SECTION C

*Q26–31 are short answer (SA) questions, 3 marks each.*

**26.** On A = {1, 2, 3, 4, 5}, the relation R = {(x, y) : $|x^2 - y^2| < 8$}. Check whether R is reflexive, symmetric and transitive. **3**

**ANSWER**

R is reflexive and symmetric but not transitive.

Reflexive: $|x^2 - x^2| = 0 < 8$ for every $x \in A$, so $(x, x) \in R$. ✓

Symmetric: if $|x^2 - y^2| < 8$ then $|y^2 - x^2| = |x^2 - y^2| < 8$, so $(y, x) \in R$. ✓

Transitive: $(1, 2) \in R$ since $|1 - 4| = 3 < 8$; $(2, 3) \in R$ since $|4 - 9| = 5 < 8$; but $|1 - 9| = 8$ is not $< 8$, so $(1, 3) \notin R$. ✗

Hence R is reflexive, symmetric, not transitive.

**OR**

$f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = ax + b$ with $f(1) = 1$ and $f(2) = 3$. Find $f(x)$ and check whether f is one-one and onto.

**ANSWER**

$f(x) = 2x - 1$; f is one-one and onto (bijective).

$a + b = 1$ and $2a + b = 3 \Rightarrow a = 2, b = -1$, so $f(x) = 2x - 1$.

One-one: $2x_1 - 1 = 2x_2 - 1 \Rightarrow x_1 = x_2$. ✓

Onto: for any $y \in \mathbb{R}$ take $x = (y + 1)/2 \in \mathbb{R}$; then $f(x) = 2 \cdot (y + 1)/2 - 1 = y$. ✓

So f is both one-one and onto.

**27.** Given $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, prove that $\dfrac{dy}{dx}=\sqrt{\dfrac{1-y^2}{1-x^2}}$. **3**

**ANSWER**

Proved: $\sin^{-1}x-\sin^{-1}y=2\cot^{-1}a \Rightarrow y'=\sqrt{(1-y^2)/(1-x^2)}$

Put x = sin θ, y = sin φ. Then cos θ + cos φ = a(sin θ − sin φ).

Using sum-to-product: $2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=2a\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$.

Cancel ⇒ $\cot\frac{\theta-\phi}{2}=a \Rightarrow \theta-\phi=2\cot^{-1}a$ (a constant).

So $\sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$. Differentiate: $\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx}=0$.

Hence $\dfrac{dy}{dx}=\dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

**OR**

Find dy/dx if y = (tan x)^x.

**ANSWER**

$$\frac{dy}{dx}=(\tan x)^x\left[\log\tan x+\frac{x\sec^2 x}{\tan x}\right]$$

Take logs: log y = x log(tan x).

Differentiate: $\dfrac{1}{y}\dfrac{dy}{dx}=\log\tan x+x\cdot\dfrac{\sec^2 x}{\tan x}$.

So $\dfrac{dy}{dx}=(\tan x)^x\left[\log\tan x+\dfrac{x\sec^2 x}{\tan x}\right]$ (the bracket can also be written log tan x + 2x cosec 2x).

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**28.** Find $\displaystyle\int\frac{x^2}{(x^2+4)(x^2+9)}\,dx$. **3**

**ANSWER**

$$-\frac{2}{5}\tan^{-1}\frac{x}{2}+\frac{3}{5}\tan^{-1}\frac{x}{3}+C$$

Put x² = t only for partial fractions: $\dfrac{t}{(t+4)(t+9)}=\dfrac{A}{t+4}+\dfrac{B}{t+9}$.

t = −4: A = −4/5; t = −9: B = 9/5.

So integrand = $-\dfrac{4}{5}\cdot\dfrac{1}{x^2+2^2}+\dfrac{9}{5}\cdot\dfrac{1}{x^2+3^2}$.

Integrate: $-\dfrac{4}{5}\cdot\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}+\dfrac{9}{5}\cdot\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}+C=-\dfrac{2}{5}\tan^{-1}\dfrac{x}{2}+\dfrac{3}{5}\tan^{-1}\dfrac{x}{3}+C$.

**OR**

Evaluate $\displaystyle\int_1^3\left(|x-1|+|x-2|+|x-3|\right)dx$.

**ANSWER**

5

On [1, 3]: |x − 1| = x − 1 and |x − 3| = 3 − x, so their sum is 2.

|x − 2| = 2 − x on [1, 2] and x − 2 on [2, 3].

Integral = $\int_1^3 2\,dx+\int_1^2(2-x)\,dx+\int_2^3(x-2)\,dx=4+\frac{1}{2}+\frac{1}{2}=5$.

**29.** Find the particular solution of $x^2\dfrac{dy}{dx} - xy = x^2\cos^2\left(\dfrac{y}{2x}\right)$, given y = π/2 when x = 1. **3**

**ANSWER**

$2\tan\left(\dfrac{y}{2x}\right) = \log|x| + 2$

Divide by x²: $\dfrac{dy}{dx} = \dfrac{y}{x} + \cos^2\dfrac{y}{2x}$ (homogeneous).

Put y = vx: $v + x\dfrac{dv}{dx} = v + \cos^2\dfrac{v}{2} \Rightarrow \sec^2\dfrac{v}{2}\,dv = \dfrac{dx}{x}$.

Integrate: $2\tan\dfrac{v}{2} = \log|x| + C \Rightarrow 2\tan\dfrac{y}{2x} = \log|x| + C$.

x = 1, y = π/2: 2 tan(π/4) = 0 + C ⇒ C = 2.

Particular solution: $2\tan\dfrac{y}{2x} = \log|x| + 2$.

**30.** Solve graphically: maximise Z = 500x + 300y subject to x + 2y ≤ 12, 2x + y ≤ 12, 4x + 5y ≥ 20, x ≥ 0, y ≥ 0. **3**

**ANSWER**

Max Z = 3200 at x = 4, y = 4

Draw the lines x + 2y = 12, 2x + y = 12 and 4x + 5y = 20. The feasible region is bounded, lying below the first two lines and above 4x + 5y = 20 in the first quadrant.

Corner points: (5, 0), (6, 0), (4, 4) [intersection of x + 2y = 12 and 2x + y = 12], (0, 6), (0, 4).

Z values: (5,0) → 2500; (6,0) → 3000; (4,4) → 3200; (0,6) → 1800; (0,4) → 1200.

Maximum Z = 3200 at (4, 4).

**31.** E and F are independent events with P($\bar{E}$) = 0.6 and P(E ∪ F) = 0.6. Find P(F) and P($\bar{E} \cup \bar{F}$). **3**

**ANSWER**

P(F) = 1/3; P($\bar{E} \cup \bar{F}$) = 13/15

P(E) = 1 − 0.6 = 0.4.

Independent ⇒ P(E ∩ F) = 0.4 P(F).

P(E ∪ F) = P(E) + P(F) − P(E ∩ F): 0.6 = 0.4 + P(F) − 0.4P(F) ⇒ 0.6 P(F) = 0.2 ⇒ P(F) = 1/3.

P($\bar{E} \cup \bar{F}$) = 1 − P(E ∩ F) (De Morgan) = 1 − 0.4 × 1/3 = 1 − 2/15 = 13/15.

## SECTION D

*Q32–35 are long answer (LA) questions, 5 marks each.*

**32.** For $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, find $A^{-1}$ and use it to solve $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$. **5**

**ANSWER**

$A^{-1} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}$; $x = 0$, $y = -5$, $z = -3$

$|A| = 1(-1 - 2) + 2(2 - 0) + 0 = -3 + 4 = 1 \neq 0$, so $A^{-1}$ exists.

Cofactors: $A_{11} = -3$, $A_{12} = -2$, $A_{13} = -4$; $A_{21} = 2$, $A_{22} = 1$, $A_{23} = 2$; $A_{31} = 2$, $A_{32} = 1$, $A_{33} = 3$.

adj A (transpose of cofactor matrix) $= \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}$, and $A^{-1}$ = adj A / 1 = the same matrix.

The system is $AX = B$ with $B = [10, 8, 7]^T$ (its coefficient matrix is exactly A).

$X = A^{-1}B$: $x = -30 + 16 + 14 = 0$; $y = -20 + 8 + 7 = -5$; $z = -40 + 16 + 21 = -3$.

Check: $0 + 10 = 10$ ✓, $0 + 5 + 3 = 8$ ✓, $10 - 3 = 7$ ✓.

**OR**

Given $A = \begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix}$, find $(a + x) - (b + y)$.

**ANSWER**

$a = 1$, $x = 3$, $b = 5$, $y = -4$; $(a + x) - (b + y) = 3$

Use $AA^{-1} = I$ and compare entries.

Row 3 of A × column 1 of $A^{-1}$: $3(1) + 1(-8) + 1(b) = 0 \Rightarrow b = 5$.

Row 3 × column 2: $3(-1) + 7 + y = 0 \Rightarrow y = -4$.

Row 1 × column 3: $-1 - 5a + 6 = 0 \Rightarrow a = 1$.

Row 2 × column 3: $1 - 10 + 3x = 0 \Rightarrow x = 3$.

(Check row 1 × column 1: $-1 - 8a + 2b = -1 - 8 + 10 = 1$ ✓.)

$(a + x) - (b + y) = (1 + 3) - (5 - 4) = 3$.

**33.** Evaluate $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$. **5**

**ANSWER**

$\frac{1}{20}\log 3$

Put t = sin x − cos x ⇒ dt = (cos x + sin x)dx. Then $t^2 = 1 - \sin 2x \Rightarrow \sin 2x = 1 - t^2$.

Limits: x = 0 → t = −1; x = π/4 → t = 0.

Denominator: $9 + 16(1 - t^2) = 25 - 16t^2$.

$I = \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{(5/4)^2 - t^2} = \frac{1}{16}\cdot\frac{1}{2\cdot\frac{5}{4}}\left[\log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$.

$= \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right] = \frac{1}{40}\log 9 = \frac{1}{20}\log 3$.

**OR**

Evaluate $\int_0^{\pi/2} \sin 2x\ \tan^{-1}(\sin x)\,dx$.

**ANSWER**

$\frac{\pi}{2} - 1$

sin 2x = 2 sin x cos x. Put sin x = t, cos x dx = dt; limits 0 → 0, π/2 → 1.

$I = 2\int_0^1 t\tan^{-1} t\,dt$.

By parts (u = $\tan^{-1}$t, dv = t dt): $\int t\tan^{-1} t\,dt = \frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int \frac{t^2}{1+t^2}dt = \frac{t^2}{2}\tan^{-1} t - \frac{1}{2}(t - \tan^{-1} t)$.

At 1: $\frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$; at 0: 0.

I = 2(π/4 − 1/2) = π/2 − 1.

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**34.** Using integration, find the area of the part of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ that lies between the lines x = −2 and x = 2. **5**

**ANSWER**

$4\sqrt{3} + \frac{8\pi}{3}$ sq units

For the ellipse, y = ½√(16 − x²). The region is symmetric about both axes.

Area = $4\int_0^2 \frac{1}{2}\sqrt{16 - x^2}\,dx = 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^2$.

At x = 2: $\frac{2}{2}\sqrt{12} + 8\cdot\frac{\pi}{6} = 2\sqrt{3} + \frac{4\pi}{3}$.

Area = 2(2√3 + 4π/3) = 4√3 + 8π/3 square units.

(Sketch: ellipse with vertices (±4, 0), (0, ±2), and the strip between x = −2 and x = 2 shaded.)

**35.** The image of a point P(x, y, z) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is P′(1, 0, 7). Find P. **5**

**ANSWER**

P(1, 6, 3)

The foot of the perpendicular M from P′ (same as from P) is the mid-point of PP′ and lies on the line:

M = (λ, 2λ + 1, 3λ + 2).

Direction of P′M: (λ − 1, 2λ + 1, 3λ − 5). It is perpendicular to (1, 2, 3):

(λ − 1) + 2(2λ + 1) + 3(3λ − 5) = 0 ⇒ 14λ − 14 = 0 ⇒ λ = 1, so M = (1, 3, 5).

M is the mid-point of P and P′: (x + 1)/2 = 1, (y + 0)/2 = 3, (z + 7)/2 = 5 ⇒ x = 1, y = 6, z = 3.

P = (1, 6, 3).

## SECTION E

*Q36–38 are case-study based questions, 4 marks each.*

**36.** *Case Study 1: Traffic police use speed cameras that can catch a speeding vehicle from up to 300 m away. One camera is fixed on a pole 5 m high, and a car C moves away from the pole at 20 m/s. When the car is x m from the foot of the pole, the angle of elevation of the camera from the car is θ.* **4**

(i) Express θ in terms of the height of the camera and x. (1)

(ii) Find dθ/dx. (1)

(iii)(a) Find the rate of change of the angle of elevation with respect to time when the car is 50 m from the pole. (2)

**ANSWER**

(i) $\theta = \tan^{-1}(5/x)$ (ii) $d\theta/dx = -5/(25 + x^2)$ (iii) $d\theta/dt = -4/101$ rad/s

(i) $\tan\theta = \text{height/distance} = 5/x \Rightarrow \theta = \tan^{-1}(5/x)$.

(ii) $$\frac{d\theta}{dx} = \frac{1}{1 + 25/x^2} \cdot \left(-\frac{5}{x^2}\right) = -\frac{5}{x^2 + 25}.$$

(iii) $$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt} = -\frac{5}{25 + x^2} \times 20.$$ At x = 50: −100/2525 = −4/101 rad/s.

The negative sign means the angle decreases as the car moves away.

**OR**

(iii) (b) (iii)(b) For another car 50 m from the foot of the pole, the angle of elevation is changing at the rate of 3/101 rad/s (in magnitude). Find the speed of this car. (2)

**ANSWER**

Speed = 15 m/s

$$\frac{d\theta}{dt} = -\frac{5}{25 + x^2} \cdot \frac{dx}{dt}.$$

At x = 50: 25 + 2500 = 2525, so magnitude of $d\theta/dt = \frac{5}{2525}\left|\frac{dx}{dt}\right| = \frac{1}{505}\left|\frac{dx}{dt}\right|$.

Set equal to 3/101: $|dx/dt| = 3 \times 505/101 = 15$.

Speed of the car = 15 m/s.

**37.** *Case Study 2: Climate change has made air turbulence more common. An airplane meets severe, moderate or light turbulence with equal probability, and the chance of arriving late is 55%, 37% and 17% respectively for these three.* **4**

(i) Find the probability that an airplane reaches its destination late. (2)

(ii) Given that the airplane reached late, find the probability that this was due to moderate turbulence. (2)

**ANSWER**

(i) 109/300 (ii) 37/109

Let $E_1$, $E_2$, $E_3$ = severe, moderate, light turbulence; $P(E_i) = 1/3$ each. A = reaching late.

$P(A|E_1) = 0.55$, $P(A|E_2) = 0.37$, $P(A|E_3) = 0.17$.

(i) Total probability: $P(A) = (1/3)(0.55 + 0.37 + 0.17) = (1/3)(1.09) = 109/300$.

(ii) Bayes' theorem: $P(E_2|A) = \dfrac{\frac{1}{3} \times 0.37}{109/300} = \dfrac{37/300}{109/300} = \dfrac{37}{109}$.

**38.** *Case Study 3: A function has an inverse only when it is one-one and onto. Sine on $\mathbb{R}$ is neither (its graph is the familiar wave between −1 and 1), so sine is restricted to an interval A, mapped onto [−1, 1], so that $\sin^{-1} : [-1, 1] \to A$ exists.* **4**

(i) Give an example of such an interval A other than the principal value branch. (1)

(ii) Using the principal value branch, find $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$. (1)

(iii)(a) Draw the graph of $\sin^{-1}x$ on [−1, 1] for its principal value branch. (2)

**ANSWER**

(i) e.g. [π/2, 3π/2] (ii) −2π/3 (iii) increasing curve from (−1, −π/2) through (0, 0) to (1, π/2)

(i) Any interval of length π on which sine is one-one and takes all values in [−1, 1], e.g. [π/2, 3π/2] or [−3π/2, −π/2].

(ii) $\sin^{-1}(-1/2) = -\pi/6$ and $\sin^{-1}(1) = \pi/2$, so the value $= -\pi/6 - \pi/2 = -2\pi/3$.

(iii) Domain [−1, 1], range [−π/2, π/2]. Plot points (−1, −π/2), (−1/2, −π/6), (0, 0), (1/2, π/6), (1, π/2) and join them with a smooth increasing curve (the mirror image of y = sin x on [−π/2, π/2] in the line y = x).

**OR**

(iii) (iii)(b) Find the domain and range of $f(x) = 2\sin^{-1}(1 - x)$. (2)

(b)

**ANSWER**

Domain [0, 2]; range [−π, π]

$\sin^{-1}$ needs $-1 \le 1 - x \le 1 \Rightarrow -2 \le -x \le 0 \Rightarrow 0 \le x \le 2$. Domain = [0, 2].

$-\pi/2 \le \sin^{-1}(1 - x) \le \pi/2 \Rightarrow -\pi \le 2\sin^{-1}(1 - x) \le \pi$. Range = [−π, π].

**— END —**